PFAFF SYSTEMS THEORY AND THE UNIFICATIONS OF GRAVITATION AND ELECTROMAGNETISM

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Abstract. We show in the framework of Pfaff systems theory, the functional dependences of the general analytic solutions of a suitable system of involutive differential equations describing the differences between the analytic solutions of the conformal and "Poincaré" Lie equations. Then we ascribe to the infinitesimal variations of the parametrizing functionals some physical meanings as the electromagnetic and gravitation potentials. We also deduce their corresponding fields of interactions together with the differential equations they must satisfy. Then we discuss on various possible physical interpretations.

1. Introduction

In this paper, we present results about smooth deformations by parametrizing functionals of the general solutions of the conformal Lie equations and propose a model as well as suggestions for a unification of electromagnetism and gravitation. This unification also has its roots, first in the conformal Lie structure that has extensively been studied first, by J. Gasqui [6] and J. Gasqui & H. Goldschmidt [8], and second, in the non-linear cohomology of Lie equations studied by H. Goldschmidt & D. Spencer [9, see references therein]. Meanwhile we only partially refer to some of its aspects since it concerns meanly the general theory of Lie equations developed by B. Malgrange [15], A. Kumpera & D. Spencer [12] and not exactly the set of PDE's we present. Indeed these latter are not the conformal Lie equations themselves but a kind of "residue" coming from the comparison with the "Poincaré Lie equations". Consequently we use the framework of Pfaff systems theory rather than the Spencer theory of Lie equations. The results we give in this paper, can be obtained in an equivalent way by using the Spencer theory but in a very cumbersome way [19]. The Pfaff systems theory is lighter and is used in the context of solutions of PDE's given by formal series. It is from our opinion completely equivalent but much more simple. In fact we think that the procedure we develop is similar to the one presented by I. G. Lisle, G. J. Reid & A. Boulton [16]. In fact, it is a Spencer theory but without its complex and boring terminology and concepts, together with the inherent difficulties coming when applied. Furthermore, this work is the result of informal reflections about an increasing amount of contradictions and incoherencies

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concerning mainly the concept of relativistic interaction, (that we find more and more serious) in the field of quantum physics as well as in classical physics. We refer the reader to a description of these contradictions in relations to F. Lurçat's [14], J.-M. Lévy-Leblond's works [13] in chapter 2. The latter gives also the main goals of our approach and our initial motivations in the field of solid state physics and in particular in the anyons theory in high-T_c superconductors. It can be over-reading at a first glance, although we give a particular spotlight on links between unifications of forces and physics of crystals, and we think it is unusual.

In the chapter 3, we present well-known results of the conformal Lie structures that are necessary for our purpose. Therein we give the set of systems of PDE's from which we start our study. It can be again over-reading by experts of the conformal geometries, since we essentially give the origins of the various PDE's we consider. They were obtained and detailed in H. Weyl [23], K. Yano [25] and J. Gasqui & H. Goldschmidt works [8]. No new results are given concerning the conformal and Poincaré Lie structures. In fact, in this presentation, our goal is meanly to highlight the relations between these two kind of structures, as well as their links with physics of gravitation.

Chapter 4, the core of this paper, begins with the formal series and the Pfaff systems of 1-forms we need and from which the functional dependences of the general analytic solutions appear, as well as their physical meanings. This formalism can be viewed as as kind of Spencer theory or a differential rings theory. Then we can build up differential sequences similar to the Spencer ones, but well-known in Pfaff systems theory in contrary to the latter.

Chapter 5 deals with the construction of the differential sequences. Then in the last chapters we conclude with suggestions for applications of these results to unifications of interactions and cosmologic considerations.

2. Goals and problems: The physics of crystals and a relativistic phenomenology of anyons

Our initial motivation shall be seen as extremely far from the problems with unifications. Actually, we were more concerned in a simple minor model of a relativistic phenomenology of creation of anyons, accurate for certain crystals [20]. At the origin of this process of creation, we suggested the kinetico-magnetoelectrical effect as described by E. Asher [2] and which has its roots in the former Minkowski works about the relations between tensors of polarization $\mathcal{P}$ and Faraday tensors $\mathcal{F}$ in a moving material of optical index $n \neq 1$. These relations can be established by turning the following diagram into a commutative one:

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\Lambda} & \mathcal{F} \\ {\scriptstyle \Upsilon'}\downarrow & & \downarrow{\scriptstyle \Upsilon} \\ \mathcal{P}' & \xrightarrow[\Lambda]{} & \mathcal{P} \end{array}$$

where Λ is a Lorentz transformation, allowing us to shift from a frame R' to a frame R, and Υ' and Υ are respectively the tensors of susceptibility within those two frames, as well supposing $\mathcal{P}$ (or $\mathcal{P}'$) linearly depending on $\mathcal{F}$ (respectively $\mathcal{F}'$). Resulting from this commutativity, the tensor Υ linearly depends on Υ' in general and also on a velocity 4-vector U associated to Λ (*e.g.* the relative velocity 4-vector between R and R'). In assimilating R' to the moving crystal frame and R to the laboratory frame, then to an applied electromagnetic field $\mathcal{F}$ fixed in R, corresponds in R' a field of polarization $\mathcal{P}$ which varies in relation to U. This is the so-called *"kinetico-magnetoelectrical effect"*.

Parallel to this phenomenon, A. Janner & E. Asher studied the concept of relativistic point symmetry in polarized crystals [10]. Such a symmetry is defined, on the one hand, by a given discrete group G, sub-group of the so-called Shubnikov group $O(3)1'$ (where $1'$ denotes the time inversion) associated with the crystal, and on the other hand, as satisfying the following properties: to make this relativistic symmetry exists, there must be a $H(\mathcal{P})$ non-trivial group of Lorentz transformations depending on $\mathcal{P}$, for which G is a normal sub-group, and that leaves the tensor of polarization $\mathcal{P}$ invariant. In other words, if $N(G)$ is the normalizer of G in the Lorentz group $O(1,3)$, and $K(\mathcal{P})$ the sub-group of $O(1,3)$ leaving $\mathcal{P}$ invariant, then $H(\mathcal{P})$ is the maximal sub-group such that:

$$\begin{aligned} &H(\mathcal{P}) \subseteq K(\mathcal{P}) \cap N(G)\,, \\ &H(\mathcal{P}) \cap O(3)1' = G\,. \end{aligned}$$

We can prove that $H(\mathcal{P})$ is about to exist only if a particular non-vanishing set V of velocity 4-vectors, invariant by the action of G, is present and consequently compatible with a kinetico-magnetoelectrical effect [2]. Therefore, if there is an interaction between moving particles in the crystal and the polarization $\mathcal{P}$, then the trajectories and $\mathcal{P}$ are obviously modified, and so is $H(\mathcal{P})$. In this process, only the group $N(G)$ is conserved so that the polarization and the trajectories are deducible during the time by the action of $N(G)$.

As we shall stipulate later on, the existence of an interaction will emerge due to a correlation between the position 3-vectors $\vec{r}$ of the charge carriers and a particular 3-vector $\vec{w}$ ($\notin V$ in general) associated with $\mathcal{P}$; $\vec{w}$ becoming then a function of $\vec{r}$. In order to allow a cyclotron-type motion which is implicit within the theory of anyons, the group $N(G)$ must contain the group $SO(2)$ and the latter must also non-trivially act on all the groups $H(\mathcal{P})$ associated to G. Then, only 12 groups G are compatible with such a description [20, 21, see the table therein and the 12 groups for which toroidal phases exist]:

$$1,\ 2',\ m,\ m',\ \bar{1}',\ 2'/m,\ \bar{3}',\ 2/m',\ 4/m',\ 6/m',\ \bar{4}',\ \bar{6}'\,.$$

It has to be noticed that among these symmetries allowing toroidal moments in crystals, most of them impose coupling of pairs of toroidal moments, each one associated to a carrier, to avoid a non-vanishing total orbital moment not compatible with the symmetry. Then, a coupling of charge carriers can occur without the need of any kind of particular (always unknown) interactions ! It can be viewed as an alternative to BCS type couplings. We can speak about a kind of *"inverse kineto-magnetoelectric effect"* since the crystals are motionless in contrary to the carriers.

In fact, throughout this development, we implicitly use a principle of equivalence similar to the one formulated in general relativity: one cannot distinguish a cyclotron-type motion in a constant polarization field from a uniform rectilinear motion in a field of polarization varying in time by action of the normalizer $N(G)$. From an other point of view, the interaction is considered to allow the extension of an invariance with respect to $H(\mathcal{P})$ to an invariance with respect to $N(G)$. The lack of interaction is then what breaks down the symmetry !

This type of reasoning concerns in fact a large amount of physical phenomena such as the spin-orbit interaction for instance. In this context, the cyclotron-type motion of electrons in anyonic states would be similar to the Thomas or Larmor precessions (see also the Coriolis or Einstein-Bass effects). More precisely, taking up again a computation, analogous to the Thomas precession one [3] (*e.g.* considering as a constant the scalar product of two tangent vectors being two parallel transports along the trajectory [5]), concerning a charge carrier at $\vec{r}$ with the velocity 4-vector U together in R, "polarized" by $\vec{w}(\vec{r})$ such as for example ($\xi = (0,\vec{\xi})_R$ constant and $\vec{\xi} \in V$):

$$W = (0,\vec{w})_R \equiv -P\,.\,\xi \text{ or } {}^*P\,.\,\xi\,,$$

where $\mathcal{P}$ depends on $\vec{r}$, one can prove from $\mathcal{P}\,.\,\xi\,.\,U \equiv W\,.\,U = cst$ (t being the laboratory frame time and ($\tilde{r} = (t,\vec{r})_R$) that:

$$\frac{dU}{dt} = (-e/m) F_{eff.}(\tilde{r})\,.\,U\,, \tag{1}$$

where m and e are respectively the mass and the electric charge of the carrier and $F_{eff.}(\tilde{r}) \equiv (\vec{E}_{eff.}(\tilde{r}), \vec{B}_{eff.}(\tilde{r}))$ is an effective Faraday tensor such that ($\gamma = (1-\vec{w}^2)^{-1/2}$ and $\vec{j} = e\, d\vec{r}/dt$):

$$\vec{B}_{eff.}(\tilde{r}) = (m/e)\left(\frac{\gamma}{1+\gamma}\right)\vec{w}\wedge\frac{d\vec{w}}{dt} \equiv (m/e^2)\left(\frac{\gamma}{1+\gamma}\right)\vec{w}\wedge\left[\vec{j}.\vec{\nabla}\right]\vec{w}\,,$$
$$\vec{E}_{eff.}(\tilde{r}) = \vec{0}\,.$$

In fact we have just taken the general formula for the Thomas precession, and then substituting the velocity 3-vector by $\vec{w}$ and the acceleration 3-vector by the time derivative of $\vec{w}$ depending on the space position 3-vector $\vec{r}$. Clearly, $F_{eff.}$ is an element of the Lie algebra of the group $SO(2)$ included in $N(G)$ and with $\vec{B}_{eff.} \in V$. Therefore this magnetic field $\vec{B}_{eff.}$ or $\vec{\xi}$ (up to a constant) might be considered as the effective magnetic field of the flux-tube V generating the so-called Aharanov-Böhm effect at the origin of the statistical parameter in the anyons theory [24]. Let us add that in general the divergence $\mathrm{Div}(\vec{B}_{eff.}) \neq 0$ so that one gets a non-vanishing density of effective magnetic monopoles generated by the local variations (due to the interaction) of the polarization vector field $\vec{w}$ in the crystal. Thus an anyon would be an effective magnetic monopole associated with a charge carrier, namely a dyon. Moreover, because this effective Faraday tensor is no more a closed two-from, a non-vanishing Chern-Simon has to be taken into account in a Lagrangian description of anyons, from which non-vanishing spontaneous constant currents can occur. On that subject, one can notice that the equation (1) can be rewritten in an orthonormal system of local coordinates ($i, j, k = 1, ..., n$; the Γ's being skew-symmetric connection symbols):

$$\dot{U}^i + \Gamma^i_{j,k}\, U^j\, U^k = 0\,. \tag{2}$$

We recognize the equations of the geodesics associated to a Riemannian connection with torsion which is thus associated to a *"generalized Thomas precession"* phenomena. This would suggest a unification in reference to the Einstein-Cartan theory but as we shall see it is not the case despite the appearances. Then if we keep on with the assumption that one has to add to the electromagnetic field a gravitational field and that the derivatives of the fields are functions of the fields themselves (as with the Bianchi identities according to the non-abelian theory for example), that means we make the assumption of the existence of a differential sequence. In electromagnetism, it is a matter of the de Rham sequence but gravitation does not interfere. The sequence integrating the latter - and being the purpose of this paper - might be a certain generalizing complex like the Spencer one.

3. The conformal finite Lie equations

First of all, let us assume that the group of relativity is not the Poincaré group anymore but the conformal Lie group (we know from Bateman and Cunningham studies [4] that it is the group of invariance of the Maxwell equations). In particular, this involves that no changes occur shifting from a given frame to a uniformly accelerated relative one. From a historical point of view, that happened to be the starting point of the Weyl theory which was finally in contradiction with experimental data.

Let us first call $\mathcal{M}$, the base space (or space-time), assumed to be of class C^∞, of dimension $n \geq 4$, connected, paracompact, without boundaries, oriented and endowed with a metric 2-form ω, symmetric, at least of class C^2 on $\mathcal{M}$ and non-degenerated but not necessarily definite positive. We also assume $\mathcal{M}$ to have a constant Riemannian scalar curvature. From these considerations, $\mathcal{M}$ has a pseudo-Riemannian or Riemannian structure because of these metric features.

The conformal finite Lie equations are deduced from the conformal action on the metric defining a pseudo-Riemannian structure on $\mathcal{M}$. Let us consider $\hat{f} \in Diff^1_{\text{loc.}}(\mathcal{M})$, the set of local diffeomorphisms of $\mathcal{M}$ of class C^1, and any function $\alpha \in C^0(\mathcal{M}, \mathbb{R})$. Then if $\hat{f} \in \Gamma_{\widehat{\mathrm{G}}}$ ($\Gamma_{\widehat{\mathrm{G}}}$ being the pseudogroup of local conformal bidifferential maps on $\mathcal{M}$), $\hat{f}$ is a solution of the following system of PDE's (in fact other PDE's must be satisfied to completely define $\Gamma_{\widehat{\mathrm{G}}}$ as one shall see in the sequel):

$$\hat{f}^*\omega = \mathrm{e}^{2\alpha}\omega \tag{3}$$

with $\det(J(\hat{f})) \neq 0$, and where $J(\hat{f})$ is the Jacobian of $\hat{f}$, and $\hat{f}^*$ is the pull-back of $\hat{f}$. Also only the $\mathrm{e}^{2\alpha}$ positive functions are taken into account because of the previous assumption that only one orientation is chosen and kept on $\mathcal{M}$, therefore we consider only the $\hat{f}$'s keeping the orientation. We have to recall that α is a varying function depending on each $\hat{f}$. We denote $\tilde{\omega}$ the metric on $\mathcal{M}$ such as by definition: $\tilde{\omega} \equiv \mathrm{e}^{2\alpha}\omega$, and we agree to put a tilde on each tensor or geometrical "object" relative to or deduced from this metric $\tilde{\omega}$. Let us notice that the latter depends on a fixed given element $\hat{f} \in \Gamma_{\widehat{\mathrm{G}}}$ when expressed without the scalar function α.

Now, doing a first prolongation of the system (3), we deduce other second order PDE's connecting the covariant derivations of Levi-Civita ∇ and $\widetilde{\nabla}$ respectively associated to ω and $\tilde{\omega}$. These new differential equations are [6] $\forall\, X,\, Y \in T\mathcal{M}$,

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\alpha(X)Y + d\alpha(Y)X - \omega(X,Y)\,{}_*d\alpha, \tag{4}$$

where d is the exterior differential and ${}_*d\alpha$ is the dual vector field of the 1-form $d\alpha$ with respect to the metric ω, *e.g.* such as $\forall\, X \in T\mathcal{M}$,

$$\omega(X, {}_*d\alpha) = d\alpha(X)\,. \tag{5}$$

Prolonging again and using the definition of the Riemann tensor ρ associated to ω, one obtains the following relation $\forall\, X, Y \in C^1(T\mathcal{M})$, $\forall\, Z \in C^2(T\mathcal{M})$, $\omega \in C^2(S^2T^*\mathcal{M})$, $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$ and $\forall\, \hat{f} \in\ Diff^3_{\text{loc.}}(\mathcal{M})$,

$$\begin{aligned}\tilde{\rho}(X,Y).Z = \rho(X,Y).Z + \omega(X,Z)\nabla_Y({}_*d\alpha) +& \\ \{\omega(\nabla_X({}_*d\alpha), Z) + \omega(X,Z)d\alpha({}_*d\alpha)\}\, Y -& \\ \{\omega(\nabla_Y({}_*d\alpha), Z) + \omega(Y,Z)d\alpha({}_*d\alpha)\}\, X +& \\ \{d\alpha(X)\omega(Y,Z) - d\alpha(Y)\omega(X,Z)\}\,{}_*d\alpha +& \\ \{d\alpha(Y)X - d\alpha(X)Y\}\, d\alpha(Z) - \omega(Y,Z)\nabla_X({}_*d\alpha).&\end{aligned} \tag{6}$$

Then we Consider $\mathcal{M}$ to be conformally flat because of the constant Riemann scalar curvature, the Weyl tensor τ associated with ω vanishes. Hence, the Riemann tensor

ρ can be rewritten $\forall\, U \in C^0(T\mathcal{M})$, $\forall\, X, Y \in C^1(T\mathcal{M})$, $\forall\, Z \in C^2(T\mathcal{M})$ as:

$$\omega(U, \rho(X, Y).Z) = \frac{1}{(n-2)} \left\{\omega(X, U)\sigma(Y, Z) - \omega(Y, U)\sigma(X, Z) + \right. \\ \left. \omega(Y, Z)\sigma(X, U) - \omega(X, Z)\sigma(Y, U)\right\}, \quad (7)$$

where σ is what we call the "Yano tensor" (see the tensor "L" in [25] differing from σ by the fraction $1/(n-2)$, or the tensor ω in [8] formula (3.12) p.68) defined $\forall\, X, Y \in C^1(T\mathcal{M})$ by

$$\sigma(X, Y) = \rho_{\mathrm{ic}}(X, Y) - \frac{\rho_{\mathrm{s}}}{2(n-1)}\omega(X, Y), \quad (8)$$

where ρ_{ic} is the Ricci tensor and ρ_{s} is the Riemann scalar curvature. Consequently, the system of PDE (6) can be rewritten as a first order system of PDE concerning σ. In order to do this, we first define two suitable trace operators, used in the sequel to obtain the $\tilde{\rho}_{\mathrm{ic}}$ and $\tilde{\rho}_{\mathrm{s}}$ tensors and finally the $\tilde{\sigma}$ tensor. Let us denote Tr^1 the trace operator defined such that for any vector bundle E over $\mathcal{M}$ we have:

$$\mathrm{Tr}^1 : T\mathcal{M} \otimes T^*\mathcal{M} \otimes E \longrightarrow E$$

with $\mathrm{Tr}^1(X \otimes \alpha \otimes \mu) = \alpha(X)\mu$ for any $X \in T\mathcal{M}$, $\alpha \in T^*\mathcal{M}$ and $\mu \in E$. Then, the second trace operator is the natural trace Tr_ω associated to ω and defined by:

$$\mathrm{Tr}_\omega : \overset{2}{\otimes} T^*\mathcal{M} \longrightarrow \mathbb{R},$$

such that $\mathrm{Tr}_\omega(u \otimes v) = v({}_*u)$. Finally, with $\mathrm{Tr}^1\tilde{\rho} = \tilde{\rho}_{\mathrm{ic}}$ and $\mathrm{Tr}_\omega\tilde{\rho}_{\mathrm{ic}} = \tilde{\rho}_{\mathrm{s}}$, we deduce first from the relations (6) and (8) $\forall\, \hat{f} \in Diff^3_{\mathrm{loc.}}(\mathcal{M})$, $\forall\, X, Y \in C^1(T\mathcal{M})$ and $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$,

$$\tilde{\sigma}(X, Y) = \sigma(X, Y) + \\ (n-2)\Big(d\alpha(X)d\alpha(Y) - \frac{1}{2}\omega(X, Y)d\alpha({}_*d\alpha) - \omega(\nabla_X({}_*d\alpha), Y)\Big). \quad (9)$$

In fact this expression can be symmetrized and using the torsion free property of the Levi-Civita covariant derivations, one obtains $\forall\, \hat{f} \in Diff^3_{\mathrm{loc.}}(\mathcal{M})$, $\forall\, X, Y \in C^1(T\mathcal{M})$ and $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$:

$$\tilde{\sigma}(X, Y) = \sigma(X, Y) + (n-2)\Big(d\alpha(X)d\alpha(Y) - \mu(X, Y) - \frac{1}{2}\omega(X, Y)d\alpha({}_*d\alpha)\Big), \quad (10)$$

into which we define $\mu \in C^0(S^2T^*\mathcal{M})$ by:

$$\mu(X, Y) = \frac{1}{2}\left[X.\, d\alpha(Y) + Y.\, d\alpha(X) - d\alpha(\nabla_X Y + \nabla_Y X)\right]. \quad (11)$$

From the proposition 5.1 due to J. Gasqui and H. Goldschmidt [8] and the well-known theorem of H. Weyl on equivalence of conformal structures [23], and because of the Weyl tensor vanishing, the differential equation (3) is formally integrable.

Also, since the Riemann scalar curvature is assumed to be a constant, the Yano tensor satisfies the following relation (see for instance, formula (16.3) p.183 in [8]):

$$\sigma = k_0 \frac{(n-2)}{2} \omega \,, \tag{12}$$

with $\rho_{\rm s} = k_0$ ($k_0 \in \mathbb{R}$). Then, considering the system (3), the system (10) reduces to a second order system of PDE's such as $\forall\, \alpha \in C^2(\mathcal{M}, \mathbb{R})$ and $\forall\, X, Y \in C^1(T\mathcal{M})$ we have:

$$\mu(X,Y) = \frac{1}{2} \left\{ \left[k_0 \left(1 - {\rm e}^{2\alpha} \right) - d\alpha({}_*d\alpha) \right] \omega(X,Y) \right\} + d\alpha(X) d\alpha(Y) \,. \tag{13}$$

Thus, we have series of PDE's deduced from (3). In particular the system made of the PDE's (3) and (4) is formally integrable and even involutive of finite type because the symbol $\widehat{M}_3$ of order 3 is vanishing (see ${\rm g}_3^{\rm c}$ in [8]). But there are alternative versions of these PDE's in which the function $\alpha \in C^2(\mathcal{M}, \mathbb{R})$ doesn't appear. These latter are the following: from the system (3), one deduces, $\forall\, \hat{f} \in Diff^1_{\rm loc.}(\mathcal{M})$:

$$\hat{f}^* \hat{\omega} / (\det(J(\hat{f})))^{2/n} = \hat{\omega} \,, \tag{14}$$

with $\det(J(\hat{f})) \neq 0$ and $\hat{\omega} = \omega / \det(\omega)^{1/n}$, and by setting:

$${}^{\hat{f}}B(X,Y) \equiv \widetilde{\nabla}_X Y - \nabla_X Y \,,$$

where ${}^{\hat{f}}B \in T\mathcal{M} \otimes S^2 T^*\mathcal{M}$ is the second fundamental quadratic form associated to $\hat{f}$, one obtains [6] from (4) $\forall\, X, Y \in C^1(T\mathcal{M})$, $\forall\, \hat{f} \in Diff^3_{\rm loc.}(\mathcal{M})$ the second order differential equation:

$$n\, {}^{\hat{f}}B = 2\, {\rm Tr}^1({}^{\hat{f}}B) - \omega\, {}_*{\rm Tr}^1({}^{\hat{f}}B) \,. \tag{15}$$

Then the conformal Lie pseudogroup $\Gamma_{\widehat{\rm G}}$ with $\tau = 0$ is the set of functions $\hat{f} \in Diff^3_{\rm loc.}(\mathcal{M})$ satisfying the involutive system of PDE's (14) and (15). One points out again one has the supplementary differential equation (13) because of the additional assumption the Riemann scalar curvature is a constant. This differential equation is the main point allowing to build up a relative complex of smooth deformations associated to the unification model.

The Poincaré pseudogroup corresponds to the case for which $\alpha = 0$. The symbol M_2 of order 2 of this pseudogroup vanishes and therefore it is involutive, and the Poincaré pseudogroup is not formally integrable unless $\rho_{\rm s}$ is a constant [8]. In an orthonormal system of coordinates, the PDE's (3), (4) and (13) can be written, with $\det(J(\hat{f})) \neq 0$ and $i, j, k = 1, \cdots, n$ as

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{f})\ \hat{f}_i^r \hat{f}_j^s = e^{2\alpha}\omega_{ij}\ , \tag{16a}$$

$$\hat{f}_{ij}^k + \sum_{r,s=1}^{n} \gamma_{rs}^k(\hat{f})\ \hat{f}_i^r \hat{f}_j^s = \sum_{q=1}^{n} \hat{f}_q^k \left(\gamma_{ij}^q + \alpha_i\delta_j^q + \alpha_j\delta_i^q - \omega_{ij}\alpha^q\right)\ , \tag{16b}$$

$$\alpha_{ij} = \frac{1}{2}\Big\{k_0(1-\mathrm{e}^{2\alpha}) - \sum_{k=1}^{n}\alpha^k\alpha_k\Big\}\omega_{ij} + \alpha_i\alpha_j\ , \tag{16c}$$

where δ_j^i is the Kronecker tensor, and where one denotes as usual $\hat{f}_j^i \equiv \partial\hat{f}^i/\partial x^j \equiv \partial_j\hat{f}^i$, etc..., $T_k = \sum_{h=1}^n T^h\,\omega_{hk}$ and $T^k = \sum_{h=1}^n T_h\,\omega^{hk}$ for any tensor T where ω^{ij} is the inverse metric tensor, and γ is the Riemann-Christoffel form associated to ω. This is the set of our starting equations.

Definition 1. *We call the* "third order system"*, the system of PDE's for $\hat{f}_J^i$ ($|J| \leq 3$) only defined by the relations of "order 3" deduced from the first prolongation of* (16b)*. In this system, the derivatives of α do not appear. These PDE's are "shaped" like ($i,j,k,h,r = 1,\cdots,n$): $\hat{f}_{jkh}^i \equiv$ polynomial of terms $\hat{f}_K^r$ ($|K| \leq 2$) with the derivatives of the Riemann-Christoffel symbols up to order two, the metric ω, the constants n and k_0 as coefficients.*

Under a change of coordinates with a conformal application $\hat{f}$, the function $\alpha(\equiv\tilde{\alpha}_0)$ and the tensor $\tilde{\alpha}_1 \equiv \{\alpha_1,\cdots,\alpha_n\}$ are transformed onto "primed" functions and tensors such as ($j = 1,\cdots,n$):

$$\alpha'(\hat{f}) = \alpha - \frac{1}{n}\ln|\det J(\hat{f})|\,, \tag{17a}$$

$$\sum_{i=1}^{n}\hat{f}_j^i\,{\alpha'}_i(\hat{f}) = \alpha_j - \sum_{k,l=1}^{n}\frac{1}{n}\,\hat{f}_l^{-1\,k}(\hat{f})\,\hat{f}_{kj}^l\,, \tag{17b}$$

which shows essentially the affine feature of these "geometrical objects", and in particularly that the tensor $\tilde{\alpha}_1$ is associated to the second order of derivation of $\hat{f}$. Then it could be considered as an acceleration tensor. In that case, a change of acceleration would keep conformal physic laws invariant. It would be closed to Einsteinian relativity.

From a mathematical point of view, it is important to notice that μ or equivalently the tensor $\tilde{\alpha}_2 \equiv \{\alpha_{ij}, i,j = 1,\cdots,n\}$ might be considered as an Abraham-Eötvös type tensor [1], leading to a first physical interpretation (up to a constant for units and with $n = 4$) of $\tilde{\alpha}_1$ as the acceleration 4-vector of gravity and α the Newtonian potential of gravitation. On the other hand, α being associated with the dilatations perhaps it might be considered as a relative Thomson type temperature (again up to a constant for units):

$$\alpha = \ln(T_0/T)\,,$$

where T_0 is a constant temperature of reference associated with the base space-time $\mathcal{M}$. A first question arises about this temperature T_0: can we consider it as the 2.7 K cosmic background temperature ? Also from the transformation law of this function, it would involve the temperature T would not be a conformal invariant but only a Lorentz (relativistic) invariant since in that case $|\det J(\hat{f})| = 1$, a question which seems always to be under investigations for instance in "hot QCD" theories although low temperatures (*e.g.* low energies) are considered as relativistic invariants.

In fact, α could be considered either as a Newtonian potential of gravitation or a temperature (if it is physically available), or more judiciously as a sum of the two. It appears that physical interpretations could be made at two levels: a "global" one at universe scales, and a "local" one in considering forces of gravitation. In the same way, a second question arises at the "global" level about the meaning of the tensor $\tilde{\alpha}_1$ up to units: since it might be an acceleration, would it be the acceleration of the inflation process ? Then this acceleration would describe the cosmic temperature background evolution and perhaps might be equivalently associated to a cosmic (repulsive ?) background of radiation of gravitational waves. As we shall see, the various dynamics will depend only on the physical interpretations of these two parameters.

4. The functional dependence

Now we look on formal series, solutions of the system of PDE's (16). We know these series will be convergent in a suitable open subset because the system is involutive. Nevertheless we need of course to know the Taylor coefficients. For instance we can choose for the applications $\hat{f}$ and the functions α the following series at a point $x_0 \in \mathcal{M}$:

$$\hat{f}^i(x): \quad S^i(x, x_0, \{\hat{a}\}) = \sum_{|J|\geq 0}^{+\infty} \hat{a}^i_J (x - x_0)^J / |J|!\,,$$

$$\alpha(x): \quad s(x, x_0, \{c\}) = \sum_{|K|\geq 0}^{+\infty} c_K (x - x_0)^K / |K|!\,,$$

with $x \in U_{x_0} \subset \mathcal{M}$ being a suitable open neighborhood of x_0 to insure the convergence of the series, $i = 1, \cdots, n$, J and K are multiple index notations such as $J = (j_1, \cdots, j_n)$, $K = (k_1, \cdots, k_n)$ with $|J| = \sum_{i=1}^n j_i$ and similar expressions for $|K|$, $\{\hat{a}\}$ and $\{c\}$ are the sets of Taylor coefficients and $\hat{a}^i_J$ and c_K are real values and not functions of x_0, though of course there are values of functions at x_0.

4.1. **The "c" system.** We call the "c" system, the system of PDE's (16c). It is from this set of PDE's the potentials and fields of interactions could occur. From the series s, at zero-th order one obtains the algebraic equations ($i,j=1,\cdots,n$; $\boldsymbol{c}_1=\{c_1,\cdots,c_n\}$):

$$c_{ij}=\frac{1}{2}\Big\{k_0(1-\mathrm{e}^{2c_0})-\sum_{k,h=1}^{n}\omega^{kh}(x_0)c_hc_k\Big\}\omega_{ij}(x_0)+c_ic_j\equiv F_{ij}(x_0,c_0,\boldsymbol{c}_1)\,, \tag{18}$$

and it follows the c_K's such that $|K|\geq 2$, will depend recursively only on x_0, c_0 and $\boldsymbol{c}_1$. It is none but the least the meaning of involution of so-called involutive systems and this recursion property can be also related by analogy to Painlevé tests. Hence the series for α can be written as $s(x,x_0,c_0,\boldsymbol{c}_1)$. By varying x_0, c_0 and $\boldsymbol{c}_1$, we can change or not the function α. Let J_1 be the 1-jets affine bundle of the C^∞ real valued functions on $\mathcal{M}$. Then it exists a subset associated to $\boldsymbol{c}_0^1\equiv(x_0,c_0,\boldsymbol{c}_1)$, we denote $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$, of elements $({x'}_0,{c'}_0,{\boldsymbol{c}'}_1)\subset J_1$, such that there is an open neighborhood $U(\boldsymbol{c}_0^1)\subset\mathcal{S}_c^1(\boldsymbol{c}_0^1)$, projecting on $\mathcal{M}$ in a neighborhood of a given $x\in U_{x_0}$, for which for all $({x'}_0,{c'}_0,{\boldsymbol{c}'}_1)\in U(\boldsymbol{c}_0^1)$ then $s(x,x_0,c_0,\boldsymbol{c}_1)=s(x,{x'}_0,{c'}_0,{\boldsymbol{c}'}_1)$. Is this subset $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ a submanifold of J_1, involving for a fixed x, the variation ds with respect to x_0, c_0 and $\boldsymbol{c}_1$ is vanishing ? From $ds\equiv 0$ it follows that ($k=1,\cdots,n$):

$$\sigma_0\equiv dc_0-\sum_{i=1}^{n}c_i dx_0^i=0\,,$$
$$\sigma_k\equiv dc_k-\sum_{j=1}^{n}F_{kj}(x_0,c_0,\boldsymbol{c}_1)\,dx_0^j=0\,.$$

We recognize a Pfaff system, we denote P_c, generated by the 1-forms σ_0 and σ_k, and the meaning of their vanishing, *e.g.* the solutions α do not change for such variations of c_0, $\boldsymbol{c}_1$ and x_0. Also, as it can be easily verified, the Pfaff system P_c is integrable since the Fröbenius condition is satisfied, and all the prolongated 1-forms σ_K ($K\geq 2$) will be linear combinaisons of these $n+1$ generating forms from the recursion property of involution. Then the subset $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ of dimension n containing a particular element $\boldsymbol{c}_0^1\equiv(x_0,c_0,\boldsymbol{c}_1)$ is a submanifold of J_1 that we call by lack the "solutions submanifold of order one at $\boldsymbol{c}_0^1$". It is a particular leaf of, at least, a local foliation on J_1 of codimension $n+1$.

From the integrability of P_c, one deduces that it exists on J_1, local systems of coordinates $(x_0,\tau_0,\tau_1,\cdots,\tau_n)$ such that each leaf $\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ is a submanifold for which $\tau_0=cst$ and $\tau_i=cst$ ($i=1,\cdots,n$), involving all the series $s(x,{x'}_0,{c'}_0,{\boldsymbol{c}'}_1)$ with $({x'}_0,{c'}_0,{\boldsymbol{c}'}_1)\in\mathcal{S}_c^1(\boldsymbol{c}_0^1)$ equal a same function $s'(x,\tau_0,\boldsymbol{\tau}_1)$ ($\boldsymbol{\tau}_1\equiv\{\tau_1,\cdots,\tau_n\}$). Then the difference $s(x,x_0,c_0,\boldsymbol{c}_1)-s(x,{x'}_0,{c'}_0,{\boldsymbol{c}'}_1)$ satisfies the relation:

$$s(x,x_0,c_0,\boldsymbol{c}_1)-s(x,{x'}_0,{c'}_0,{\boldsymbol{c}'}_1)=s'(x,\tau_0,\boldsymbol{\tau}_1)-s'(x,{\tau'}_0,{\boldsymbol{\tau}'}_1)\,, \tag{19}$$

with $(i = 1, \cdots, n)$

$$\triangle_0 \tau \equiv {\tau'}_0 - \tau_0 = \int_{\boldsymbol{c}_0^1}^{{\boldsymbol{c}'}_0^1} \sigma_0 \,,$$

$$\triangle_i \tau \equiv {\tau'}_i - \tau_i = \int_{\boldsymbol{c}_0^1}^{{\boldsymbol{c}'}_0^1} \sigma_i \,.$$

Now, we consider the c's are values of differential functions ρ: $c_K = \rho_K(x_0)$, as expected for Taylor coefficients. Roughly speaking, we make a pull-back on $\mathcal{M}$, inducing a projection from the subbundle of projectable elements in T^*J_1 to $T^*\mathcal{M} \otimes_{\mathbb{R}} J_1$. Then, we set (with $\boldsymbol{\rho}_1 \equiv \{\rho_1, \cdots, \rho_n\}$ and no changes of notations for the pull-back):

$$\sigma_0 \equiv \sum_{i=1}^{n} (\partial_i \rho_0 - \rho_i)\, dx_0^i \equiv \sum_{i=1}^{n} \mathcal{A}_i \, dx_0^i \,, \tag{20a}$$

$$\sigma_i \equiv \sum_{j=1}^{n} (\partial_j \rho_i - F_{ij}(x_0, \rho_0, \boldsymbol{\rho}_1))\, dx_0^j \equiv \sum_{j=1}^{n} \mathcal{B}_{j,i} \, dx_0^j \,, \tag{20b}$$

and it follows that

$$\triangle_0 \tau = \int_{x_0}^{{x'}_0} \sum_{i=1}^{n} \mathcal{A}_i \, dx^i \,,$$

$$\triangle_i \tau = \int_{x_0}^{{x'}_0} \sum_{j=1}^{n} \mathcal{B}_{j,i} \, dx^j \,.$$

In particularly, if $\boldsymbol{c}_0^1 \in \mathcal{S}_c^1(0)$, the "null" submanifold corresponding to the vanishing solution of the third system with $\tau_0 = \tau_1 = \cdots = \tau_n = 0$, then the difference (19) involves that

$$\alpha(x) \equiv s(x, {x'}_0, {c'}_0, {\boldsymbol{c}'}_1) = s'(x, {\tau'}_0, {\boldsymbol{\tau}'}_1) = s(x, x \equiv {x''}_0, {c''}_0, {\boldsymbol{c}''}_1) \,,$$

with

$$ {\tau'}_0 = \int_{x_0}^{{x''}_0} \sum_{i=1}^{n} \mathcal{A}_i \, dx^i \,,$$

$$ {\tau'}_i = \int_{x_0}^{{x''}_0} \sum_{j=1}^{n} \mathcal{B}_{j,i} \, dx^j \,,$$

and ${\boldsymbol{c}''}_0^1 \in \mathcal{S}_c^1({\boldsymbol{c}'}_0^1)$. In particularly, since we can take ${x''}_0 \equiv x$ then

$$\alpha(x) \equiv s'\Big(x, \int_{x_0}^{x} \sum_{i=1}^{n} \mathcal{A}_i \, d{x'}^i \,, \int_{x_0}^{x} \sum_{j=1}^{n} \mathcal{B}_{j,1} \, d{x'}^j, \cdots, \int_{x_0}^{x} \sum_{j=1}^{n} \mathcal{B}_{j,n} \, d{x'}^j \Big) \,, \tag{21}$$

which shows the functional dependences of the solutions of the "c" system with respect to the functions ρ_0 and $\boldsymbol{\rho}_1$, themselves associated to the smooth infinitesimal deformations of these solutions. These smooth deformations can also be considered as smooth deformations from "Poincaré solutions" of the system (16) for which $\alpha \equiv 0$, to "conformal solutions" whatever is α.

Moreover the functions ρ, and consequently the functions ρ_0, $\mathcal{A}$ and $\mathcal{B}$, must satisfy additional differential equations coming from the integrability conditions of the Pfaff system P_c. More precisely, from the relations $d\sigma_0 = \sum_{i=1}^n dx_0^i \wedge \sigma_i$, $d\sigma_i = \sum_{j=1}^n dx_0^j \wedge \sigma_{ij}$ and

$$\sigma_{ij} = c_i\sigma_j + c_j\sigma_i - \omega_{ij}\Big\{k_0 \mathrm{e}^{2c_0}\sigma_0 + \sum_{k,h=1}^n \omega^{kh} c_h \sigma_k\Big\} \equiv \vartheta_{ij}(\boldsymbol{c}_0^1, \sigma_J; |J| \leq 1)\,, \tag{22}$$

one deduces a set of algebraic relations at x_0:

$$\mathcal{I}_{k,i} \equiv \partial_k \mathcal{A}_i - \mathcal{B}_{i,k}\,, \tag{23a}$$

$$\mathcal{I}_{k,i} = \mathcal{I}_{i,k}\,, \tag{23b}$$

$$\mathcal{J}_{k,j,i} \equiv \partial_j \mathcal{B}_{k,i} - \rho_i\,\mathcal{B}_{k,j} - \rho_j\,\mathcal{B}_{k,i} + \omega_{ij}\left\{k_0 e^{2\rho_0}\mathcal{A}_k + \sum_{r,s=1}^n \omega^{rs}\,\rho_r\,\mathcal{B}_{k,s}\right\}\,, \tag{23c}$$

$$\mathcal{J}_{k,j,i} = \mathcal{J}_{j,k,i}\,. \tag{23d}$$

Clearly, in these relations, $(\rho_0, \boldsymbol{\rho}_1)$ appears to be a set of arbitrary functions. In considering $\mathcal{F}$ and $\mathcal{G}$ as being respectively the skew-symmetric and the symmetric parts of the tensor of components $\partial_i\rho_j$, then one deduces, from the symmetry properties of the latter relations, what we call *the first set of differential equations*:

$$\partial_i \mathcal{F}_{jk} + \partial_j \mathcal{F}_{ki} + \partial_k \mathcal{F}_{ij} = 0\,, \tag{24a}$$

$$2\,\partial_j \mathcal{G}_{ki} - \partial_i \mathcal{G}_{kj} - \partial_k \mathcal{G}_{ij} = \partial_i \mathcal{F}_{kj} - \partial_k \mathcal{F}_{ji}\,, \tag{24b}$$

with

$$\mathcal{F}_{ij} = \partial_j \rho_i - \partial_i \rho_j = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i\,, \tag{25a}$$

$$\mathcal{G}_{ij} = \partial_i \rho_j + \partial_j \rho_i \equiv \partial_i \mathcal{A}_j + \partial_j \mathcal{A}_i \mod (\rho_0, \boldsymbol{\rho}_1)\,. \tag{25b}$$

The PDE's (24a) with (25a) might be the first set of *Maxwell equations*.

Then we give a few definitions to go further.

Definition 2. *We denote:*

1. $\theta_{\mathcal{M}}$ *the sheaf of rings of germs of the differential (e.g.* C^∞*) functions defined on* $\mathcal{M}$*,*
2. $\underline{J_1}$ *the sheaf of* $\theta_{\mathcal{M}}$*-modules of germs of differential sections of the 1-jet affine space bundle* J_1 *of the real valued differential functions defined on* $\mathcal{M}$*,*

3. *$\mathcal{S}_c^0 \subset \theta_{\mathcal{M}}$ the sheaf of rings of germs of solutions of the "c" system of algebraic equations* (16c) *at each point x_0 in $\mathcal{M}$ (not simultaneously at all point in $\mathcal{M}$; see remark below),*
4. *$\mathcal{S}_c^1 \subset \underline{J_1}$ the sheaf of $\theta_{\mathcal{M}}$-modules of germs of differential sections of J_1 defined by the system of algebraic equations at each point $x_0 \in \mathcal{M}$ (not everywhere, as mentioned above; $i, j, k = 1, \cdots, n$):*

$$\Big\{k_0(1 - e^{2\rho_0}) - \sum_{r=1}^{n} \rho^r \rho_r\Big\}(\partial_k \omega_{ij} - \partial_j \omega_{ik}) + 2k_0(\omega_{ik}\rho_j - \omega_{ij}\rho_k) + \sum_{\ell,h=1}^{n} \rho^h \rho^\ell (\omega_{ij}\partial_k \omega_{h\ell} - \omega_{ik}\partial_j \omega_{h\ell}) = 0\,, \quad (26)$$

satisfied by ρ_0 and $\boldsymbol{\rho}_1$ and deduced from the relations (23a) *and* (23c) *when $\mathcal{A}$ and $\mathcal{B}$ satisfy the relations* (20),
5. *$\underline{T^*\mathcal{M}}$ the sheaf of $\theta_{\mathcal{M}}$-modules of germs at each point $x_0 \in \mathcal{M}$ of global 1-forms on $\mathcal{M}$.*

Remark 1. We do not consider in this set of definitions, solutions of PDE's but solutions of algebraic equations at x_0, since a solution of a PDE's is a particular "coherent" subsheaf (graphs of solutions) for which equations (16c) are satisfied everywhere in $\mathcal{M}$, and not only at a given x_0 whatsoever. More precisely, for instance, the tensors $\mathcal{A}$ and $\mathcal{B}$ in (21) must be such solutions subsheafs of the PDE (23). Indeed x_0 may vary with these tensors being always solutions of the corresponding set of algebraic equations at every point x_0 of subsets containing the paths of integration (the paths can be "cut" and the algebraic equations must be satisfied at each new resulting initial point of integration).

Remark 2. The equations (26), associated to a particular leaf in J_1, define what we call the *"characteristic manifold"*. Each solution of (16c) satisfies this set of equations everywhere on $\mathcal{M}$.

Then, in considering the local diffeomorphisms $\wedge^k T^*\mathcal{M} \otimes_{\mathbb{R}} J_r \simeq (\{x_0\} \otimes_{\mathbb{R}} J_r) \times (\wedge^k T^*_{x_0}\mathcal{M} \otimes_{\mathbb{R}} J_r)$ with $0 \leq k \leq n$ and $r \geq 0$, we set the definitions:

Definition 3. *We define the operators:*

1. $j_1 : (x_0, \rho_0) \in \mathcal{S}_c^0 \longrightarrow (x_0, \rho_0, \rho_1 = \partial_1 \rho_0, \cdots, \rho_n = \partial_n \rho_0) \in \mathcal{S}_c^1$,
2. $D_{1,c} : \boldsymbol{\rho}_0^1 \equiv (x_0, \rho_0, \boldsymbol{\rho}_1) \in \mathcal{S}_c^1 \longrightarrow (\boldsymbol{\rho}_0^1, \sigma_0, \sigma_1, \cdots, \sigma_n) \in \underline{T^*\mathcal{M}} \otimes_{\theta_{\mathcal{M}}} \underline{J_1}$, *with $\mathcal{A}$, $\mathcal{B}$ and $\boldsymbol{\rho}_0^1$ satisfying relations (20),*

3. $D_{2,c} \; : \; (\boldsymbol{\rho}_0^1, \sigma_0, \sigma_1, \cdots, \sigma_n) \in \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1} \longrightarrow (\boldsymbol{\rho}_0^1, \zeta_0, \zeta_1, \cdots, \zeta_n) \in \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1}$, *with*

$$\zeta_0 = \sum_{i,j=1}^{n} \mathcal{I}_{i,j}\, dx_0^i \wedge dx_0^j \,,$$

$$\zeta_k = \sum_{i,j=1}^{n} \mathcal{J}_{j,i,k}\, dx_0^i \wedge dx_0^j \,,$$

and the functions $\boldsymbol{\rho}_0^1$ *and the tensors* $\mathcal{I}$, $\mathcal{J}$, $\mathcal{A}$ *and* $\mathcal{B}$ *satisfying the relations (23).*

Then from all the previous results we easily deduce:

Theorem 1. *The differential sequence*

$$0 \longrightarrow \mathcal{S}_c^0 \xrightarrow{j_1} \mathcal{S}_c^1 \xrightarrow{D_{1,c}} \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1} \xrightarrow{D_{2,c}} \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1} \,,$$

is exact, and the differential operators $D_{1,c}$ *and* $D_{2,c}$ *are* $\mathbb{R}$*-linear.*

Remark 3. We think this sequence is perhaps closed to a Spencer non-linear sequence for the deformations of the Lie structures of the "c" system. Indeed, since the system P_c is integrable, it is always, at least locally, diffeomorphic to an integrable subset of the set of Cartan 1-forms in $T^*\mathcal{M} \otimes_\mathbb{R} J_1$ associated to a particular finite Lie algebra g_c (of dimension greater or equal to $n+1$), with corresponding Lie group G_c acting (freely or not) on the left on each characteristic manifold. Then, $\mathcal{S}_c^1$ and $\mathcal{S}_c^0$ would be diffeomorphic respectively (at least locally) to a sheaf of Lie groups G_c, from which the Cartan forms would be defined with the first non-linear Spencer operator, and the isotropy subgroups sheaf. Also to $D_{2,c}$ would correspond Bianchi type equations, according to the Nijenhuis-Frölicher bracket in $T^*\mathcal{M} \otimes_\mathbb{R} J_1$, and defining the second non-linear Spencer operator.

The tensors $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$ and $\mathcal{J}$ above are given *"at the target"* and not *"at the source"*. Usually, at the k-jets affine bundles level, the source is the point x_0 and the target the point a_0. But at the sheafs level, since we consider maps and not points, the source is the *id* application at x_0, *e.g.* (x_0, id) and the target is the application $\hat{f}$ at x_0, *e.g.* $(x_0, \hat{f})$, with $\hat{a}_0 = \hat{f}(x_0)$. On the other side, from the formal series point of view, the source is x_0 together with the set of Taylor coefficients of *id* at x_0 and the target is x_0 together with $\{\hat{a}\}$. From the composition law of two applications $\hat{f} \circ \hat{g}$, we can define a left action of $\{\hat{a}\}$ on the corresponding set of Taylor coefficients of the application $\hat{g}$. Since the conformal Lie equations are involutive, then we obtain a left action of a finite set of Taylor coefficients of order less than 3. Hence the set of values taken by this finite set define a Lie group we denote $\widehat{G}_{2,x_0}$ at x_0. Moreover its left action links any source point $(x_0, \{\hat{a}'\})$ to any other $(x_0, \{\hat{a}''\})$. In fact, it indicates the property of transitivity of the conformal Lie groupoid defined by the system (16). Hence, its action on the source defines a left action of a transitive finite Lie group on the leafs of J_1 (not $\mathcal{M}$), as well as a similar transitive left action on $\widehat{G}_{2,x_0}$ itself. Then this group is diffeomorphic to a subgroup of the previously defined G_c group. It follows from the transitivity, inducing globality, and in case of a compact manifold $\mathcal{M}$, the integrals in (21) would define perhaps a deformation class in the first Spencer cohomology space of deformations of global sections from $\mathcal{M}$ to the sheaf of $\widehat{G}_2$ Lie groups.

In the "diagonal method approach" used by A. Kumpera & D. Spencer [12], formulas are given at the source by using the *Buttin formula*. It is out of our purpose and not necessary to give an equivalent formula in the present formalism. The results at the source will be equivalent to those at the target because of the transitivity.

We can give, as an example, what is this action of $\widehat{G}_2$ on the c's. From (17), one has the relations, with $\sum_{j=1}^n \hat{b}^i_j\, \hat{a}^j_k = \sum_{j=1}^n \hat{a}^i_j\, \hat{b}^j_k = \delta^i_k$:

$$\begin{aligned} c'_0 &= c_0 - \frac{1}{n} \ln|\det(\hat{a}^i_j)|\,, \\ \sum_{i=1}^n \hat{a}^i_j\, c'_i &= c_j - \sum_{k,l=1}^n \frac{1}{n}\, \hat{b}^k_l\, \hat{a}^l_{kj}\,. \end{aligned}$$

Remark 4. In view of physical interpretations, we compute the Euler-Lagrange equations of a Lagrangian density

$$\mathcal{L}(x_0, \rho_0, \boldsymbol{\rho}_1, \mathcal{A}, \mathcal{F}, \mathcal{G})\, d^n x_0\,, \tag{27}$$

with $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$ satisfying the relations (20a) and (25). We obtain what we call *the second set of differential equations*:

$$\sum_{k=1}^n \partial_k \left(\frac{\partial \mathcal{L}}{\partial \mathcal{A}_k}\right) = \left(\frac{\partial \mathcal{L}}{\partial \rho_0}\right)\,, \tag{28a}$$

$$\sum_{k=1}^n \left\{\partial_k \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{ik}}\right) + \partial_k \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}_{ik}}\right)\right\} = \frac{1}{2}\left\{\left(\frac{\partial \mathcal{L}}{\partial \rho_i}\right) - \left(\frac{\partial \mathcal{L}}{\partial \mathcal{A}_i}\right)\right\}\,. \tag{28b}$$

If we consider $\mathcal{A}$ as being the electromagnetic potential vector, $\mathcal{F}$ the Faraday tensor and $\mathcal{L}$ only depending on $\mathcal{A}$ and $\mathcal{F}$, then (24a) and (28b) are the *second set of Maxwell equations* if we denote by

$$\mathcal{J}^k_e = \left(\frac{\partial \mathcal{L}}{\partial \mathcal{A}_k}\right)\,,$$

the electric current components and

$$\mathcal{P}^{ki} = -2\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{ki}}\right)\,,$$

the polarization tensor components. Moreover the differential equations (28a) become the free divergence property of the electric current $\mathcal{J}_e$ if $\mathcal{L}$ is independent of ρ_0. It has to be noticed that no magnetic currents seem to occur.

4.2. **The "full" system.** This system is defined by the set of PDE's (16). For this system of Lie equations, we recall well-known results but in the present context. Applying the same reasoning than in the previous subsection, firstly we obtain the following results up to order two:

$$\sum_{r,s=1}^{n} \omega_{rs}(\hat{a}_0)\ \hat{a}_i^r\ \hat{a}_j^s = e^{2c_0}\omega_{ij}(x_0)\,, \tag{29a}$$

$$\hat{a}_{ij}^k + \sum_{r,s=1}^{n} \gamma_{rs}^k(\hat{a}_0)\ \hat{a}_i^r\ \hat{a}_j^s = \sum_{q=1}^{n} \hat{a}_q^k \left(\gamma_{ij}^q(x_0) + c_i\delta_j^q + c_j\delta_i^q - \omega_{ij}(x_0)c^q\right)\,, \tag{29b}$$

which clearly shows that J_1 is diffeomorphic to an embedded submanifold of the 2-jets affine bundle $J_2(\mathcal{M})$ of the $C^\infty(\mathcal{M},\mathcal{M})$ differentiable applications on $\mathcal{M}$. Secondly we get relations for the coefficients of order 3 that we only write as ($\hat{a}_1 \equiv (\hat{a}_j^i)$; $\hat{a}_2 \equiv (\hat{a}_{jk}^i)$, … , $\hat{a}_k \equiv (\hat{a}_{j_1\cdots j_k}^i)$; $\hat{\boldsymbol{a}}_0^k \equiv (\hat{a}_0,\cdots,\hat{a}_k)$):

$$\hat{a}_{jkh}^i \equiv \hat{A}_{jkh}^i(x_0,\hat{\boldsymbol{a}}_0^2)\,, \tag{30}$$

pointing out in this expression the independence from the "c" coefficients. We denote by $\widehat{\Omega}_J^i$ the Pfaff 1-forms at x_0 and $\{\hat{a}\}$ (or at the target $(x_0,\{\hat{a}\})$):

$$\widehat{\Omega}_J^i \equiv d\hat{a}_J^i - \sum_{k=1}^{n} \hat{a}_{J+1_k}^i dx_0^k\,, \tag{31}$$

and setting the $\hat{a}$'s as values of functions $\hat{\tau}$ depending on x_0 (in some way we make a pull-back on $\mathcal{M}$), we define the tensors $\hat{\kappa}$ and the differential operator $\widehat{\boldsymbol{\mathfrak{D}}}_1$ by:

$$\widehat{\Omega}_J^i \equiv \widehat{\boldsymbol{\mathfrak{D}}}_1\hat{\tau}_J^i \equiv \sum_{k=1}^{n}\left(\partial_k\hat{\tau}_J^i - \hat{\tau}_{J+1_k}^i\right)dx_0^k \equiv \sum_{k=1}^{n}\hat{\kappa}_{k,J}^i\,dx_0^k\,. \tag{32}$$

At this step, we can have a look on the corresponding 1-forms ${}^s\widehat{\Omega}$ at the source (x_0,id). Using the composition law $\hat{f}'' \equiv \hat{f}\circ\hat{f}'$ of two diffeomorphisms $\hat{f}$ and $\hat{f}'$, we deduce from the relations

$$\begin{aligned}
\hat{f}''^i_j &= \sum_{k=1}^{n} \hat{f}_k^i \circ \hat{f}'\ \hat{f}'^k_j\,,\\
\hat{f}''^i_{jk} &= \sum_{r,s=1}^{n} \hat{f}_{rs}^i \circ \hat{f}'\ \hat{f}'^r_j \hat{f}'^s_k + \sum_{u=1}^{n} \hat{f}_u^i \circ \hat{f}' \hat{f}'^u_{jk}\,,\\
\cdots &= \cdots\,,
\end{aligned}$$

and their Taylor coefficients versions, the relations for the 1-forms ${}^s\widehat{\Omega}$:

$$
\begin{aligned}
{}^s\widehat{\Omega}^k &= \sum_{i=1}^{n} \hat{b}_i^k \, \widehat{\Omega}^i \,, \\
{}^s\widehat{\Omega}_j^k &= \sum_{i=1}^{n} \hat{b}_i^k \left\{ \widehat{\Omega}_j^i - \sum_{s,r=1}^{n} \hat{a}_{js}^i \hat{b}_r^s \, \widehat{\Omega}^r \right\} \,, \\
\cdots &= \cdots \,.
\end{aligned}
$$

Expressing the latter with the 1-forms dx_0^i and $d\hat{a}_J^k$, we recover similar expressions than those given by J.-F. Pommaret in [18] p.214 for the "χ tensors" but, from our opinion, in a clearer context:

$$
\begin{aligned}
{}^s\widehat{\Omega}^i &= \sum_{j=1}^{n} \hat{b}_j^i \, d\hat{a}^j - dx_0^i \,, \\
{}^s\widehat{\Omega}_j^i &= \sum_{k=1}^{n} \hat{b}_k^i \left\{ d\hat{a}_j^k - \sum_{r,s=1}^{n} \hat{a}_{jr}^k \hat{b}_s^r d\hat{a}^s \right\} \,, \\
\cdots &= \cdots \,.
\end{aligned}
$$

These 1-forms are conformal invariant Cartan 1-forms. We think it also gives an other presentation of the *"Buttin formula"*. Then from the relations:

$$
\begin{aligned}
\mathrm{e}^{2c_0} \omega^{rs}(\hat{a}_0) &= \sum_{i,j=1}^{n} \omega^{ij}(x_0) \hat{a}_i^r \hat{a}_j^s \,, \\
\sum_{i=1}^{n} \gamma_{ik}^i &= \frac{1}{2} \sum_{i,j=1}^{n} \omega^{ij} \, \partial_k \omega_{ij} \,,
\end{aligned}
$$

we deduce, for example, the $\widehat{\Omega}_j^i$ 1-forms satisfy the relations at the target $(x_0, \hat{\boldsymbol{a}}_0^1)$:

$$
\widehat{H}_0(x_0, \hat{\boldsymbol{a}}_0^1, \widehat{\Omega}_L^k; |L| \leq 1) \equiv \sum_{i,j=1}^{n} \hat{b}_i^j \, \widehat{\Omega}_j^i + \sum_{j,k=1}^{n} \gamma_{jk}^j(\hat{a}_0) \widehat{\Omega}^k = n\sigma_0 \,. \tag{33}
$$

Similar computations show that the 1-forms σ_i can be expressed as quite long relations, linear in the $\widehat{\Omega}_J^j$ ($|J| \leq 2$) and with polynomials of the $\hat{a}_K$ ($|K| \leq 2$) and derivatives of the metric and the Riemann-Christoffel symbols taken either at x_0 or $\hat{a}_0$, as coefficients. Then, we set:

$$
\sigma_i \equiv \widehat{H}_i(x_0, \hat{\boldsymbol{a}}_0^2, \widehat{\Omega}_I^j; |I| \leq 2) \,. \tag{34}
$$

From (30) the 1-forms $\widehat{\Omega}_{jkh}^i$ are also sums of 1-forms $\widehat{\Omega}_K^r$ ($|K| \leq 2$) with the same kind of coefficients and not depending on the σ's, and we write:

$$
\widehat{\Omega}_{jkh}^i \equiv \widehat{K}_{jkh}^i(x_0, \hat{\boldsymbol{a}}_0^2, \widehat{\Omega}_K^r; |K| \leq 2) \,, \tag{35}
$$

where $\widehat{K}^i_{jkh}$ is linear in the 1-forms $\widehat{\Omega}^r_K$.

Denoting by $\boldsymbol{\mathcal{P}}_2 \subset J_2(\mathcal{M})$ the set of elements satisfying relations (29) whatever are the c's. Then the Pfaff system we denote $\widehat{P}_2$ over $\boldsymbol{\mathcal{P}}_2$ and generated by the 1-forms $\widehat{\Omega}^j_K \in T^*\mathcal{M} \otimes_{\mathbb{R}} J_2(\mathcal{M})$ in (31) with $|K| \leq 2$, is locally integrable on every neighborhood $U_{(x_0,\hat{\boldsymbol{a}}^2_0)} \subset J_2(\mathcal{M})$ since at the target $(x_0, \hat{\boldsymbol{a}}^2_0)$ we have ($|J| \leq 2$):

$$\widehat{\Theta}^i_J \equiv \widehat{\boldsymbol{\mathcal{D}}}_2 \widehat{\Omega}^i_J \equiv d\widehat{\Omega}^i_J - \sum_{k=1}^n dx^k_0 \wedge \widehat{\Omega}^i_{J+1_k} = 0\,, \tag{36}$$

together with (35).

From now we consider the "Poincaré system" with corresponding notation without the "hats". We denote by Ω^i_J the Pfaff 1-forms corresponding to this system, *e.g.* the system defined by the PDE's (16a) and (16b) with a vanishing function α. The corresponding 1-forms "σ" are also vanishing everywhere on $\mathcal{M}$ and the Ω^i_J satisfy all the previous relations but with the σ's cancelled out. Then it is easy to see the Ω^i_J 1-forms ($|J| \geq 2$) are generated by the set of 1-forms Ω^j_K ($|K| \leq 1$) and in particular we have

$$\Omega^k_{ij} = -\left\{ \sum_{r,s,h=1}^n (\partial_h \gamma^h_{ij})(a_0) \Omega^r\, a^r_i\, a^s_j + \sum_{r,s=1}^n \gamma^k_{rs}(a_0) [\, a^r_i\, \Omega^s_j + a^s_j\, \Omega^r_i\,] \right\}\,, \tag{37}$$

with $\hat{\boldsymbol{a}}^1_0 \equiv \boldsymbol{a}^1_0 \in \boldsymbol{\mathcal{P}}_1 \subset J_1(\mathcal{M})$, and $\boldsymbol{\mathcal{P}}_1$ being the set of elements satisfying relations (29a) with $c_0 = 0$. Similarly the Pfaff system we denote P_1 over $\boldsymbol{\mathcal{P}}_1$ and generated by the 1-forms Ω^j_K in (31) with $|K| \leq 1$, is locally integrable on every neighborhood $U_{(x_0,\boldsymbol{a}^1_0)} \subset \boldsymbol{\mathcal{P}}_1$ since at the $(x_0, \boldsymbol{a}^1_0)$ point we have the relations (36) with $|J| \leq 1$ together with the relations (37).

Then we have at each $(x_0, \hat{\boldsymbol{a}}^2_0) \in \boldsymbol{\mathcal{P}}_2$ the locally exact splitted sequence

$$0 \longrightarrow P_1 \xrightarrow{b_1} \widehat{P}_2 \xrightarrow{e_1} P_c \longrightarrow 0\,, \tag{38}$$

where we consider J_1 embedded in $J_2(\mathcal{M})$ as well as $\boldsymbol{\mathcal{P}}_1$ from relations (29). In this sequence a back connection b_1 and a connection $c_1 : P_c \longrightarrow \widehat{P}_2$ are such that ($|J| \leq 2$):

$$\widehat{\Omega}^i_J = \Omega^i_J + \chi^i_J(\boldsymbol{a}^2_0)\, \sigma_0 + \sum_{k=1}^n \chi^{i,k}_J(\boldsymbol{a}^2_0)\, \sigma_k\,, \tag{39}$$

with Ω^i_{jk} satisfying (37) for any given Ω^h_L with $|L| \leq 1$, and the tensors χ defined on $\boldsymbol{\mathcal{P}}_2$. They define together a back connection, and the tensors χ define a connection if they satisfy the relations:

$$\widehat{H}_0(x_0, \hat{\boldsymbol{a}}_0^1, \chi_L^k; |L| \leq 1) = n\,, \tag{40a}$$

$$\widehat{H}_0(x_0, \hat{\boldsymbol{a}}_0^1, \chi_L^{k,i}; |L| \leq 1) = 0\,, \tag{40b}$$

$$\widehat{H}_i(x_0, \hat{\boldsymbol{a}}_0^2, \chi_L^k; |L| \leq 2) = 0\,, \tag{40c}$$

$$\widehat{H}_i(x_0, \hat{\boldsymbol{a}}_0^2, \chi_L^{k,h}; |L| \leq 2) = n\delta_i^h\,, \tag{40d}$$

in order to keep the relations (33) and (34), *e.g.* $e_1 \circ c_1 = id$.

Remark 5. The corresponding *"characteristic manifolds"* for the conformal and Poincaré Pfaff systems are defined by tremendous sets of algebraic equations. We give rather a way of computations to get them if really necessary (!) but especially to see their algebraic features in the k-jets affine bundles on $\mathcal{M}$. First of all, as it can easily be verified, these algebraic equations come from the $(k+1)$th-order. For instance, in the conformal case, we begin with the equations (30) of order three since the conformal Pfaff system is defined on $J_2(\mathcal{M})$. Then we easily deduce a first set of relations of the kind:

$$\widehat{\Omega}^i_{jkh} = \sum_{r=1}^{n} \sum_{|L|\leq 2} \left(\frac{\partial \widehat{A}^i_{jkh}(x_0, \hat{\boldsymbol{\tau}}_0^2)}{\partial \hat{\tau}^r_L} \right) \widehat{\Omega}^r_L\,,$$

where we substitute the values $\hat{a}$ by the functions $\hat{\tau}$ satisfying the relations (30) again. From (32) with $|J| = 2$, we finally obtain the algebraic equations of the characteristic manifold as expressions such as

$$\sum_{j,k=1}^{n} \left\{ \left(\frac{\partial \widehat{A}^i_{rsk}}{\partial x_0^j} \right) + \sum_{u=1}^{n} \sum_{|L|\leq 2} \left(\frac{\partial \widehat{A}^i_{rsk}}{\partial \hat{\tau}^u_L} \right) \hat{\tau}^u_{L+1_j} \right\} dx_0^j \wedge dx_0^k = 0\,,$$

into which the $\hat{\tau}^u_K$ functions of order three ($|K| = 3$) are expressed by functions $\hat{\tau}$ of less order with relations (30). It underlines the skew-symmetry property of these characteristic manifolds which appear to be symmetric spaces.

5. The differential sequences

Let us denote by $\Pi_k(\mathcal{M})$ the subbundle of $J_k(\mathcal{M})$ of the local diffeomorphisms of $\mathcal{M}$, and $j_k : (x_0^j, \hat{\tau}^i) \in \underline{J_0(\mathcal{M})} \longrightarrow (x_0^j, \hat{\tau}^i, \partial_j \hat{\tau}^i, \cdots, \partial_{j_1 \cdots j_k} \hat{\tau}^i) \in \underline{J_k(\mathcal{M})}$, where we underline the k-jets affine bundles to indicate their corresponding sheafs as usual. As for the "c" system we give the following set of definitions.

Definition 4. *We denote:*

1. $\Gamma_0(\mathcal{M}) \subset \underline{\Pi_0(\mathcal{M})}$ *the sheaf of* $\theta_{\mathcal{M}}$*-modules of germs of solutions of the Poincaré system of algebraic equations* (16a) *and* (16b) *with* $\alpha = 0$ *at each point* x_0 *in* $\mathcal{M}$*,*

2. $\Gamma_1(\mathcal{M}) \subset \underline{\Pi_1(\mathcal{M})}$ *the sheaf of* $\theta_{\mathcal{M}}$*-modules of germs of differential sections of* $\Pi_1(\mathcal{M})$ *satisfying the* "characteristic manifold" *algebraic equations for the Poincaré system at each* $x_0 \in \mathcal{M}$*, and which inherits of the* "Lie groupoid structure" *from* $\Pi_1(\mathcal{M})$.

Definition 5. *We denote:*

1. $\widehat{\Gamma}_0(\mathcal{M}) \subset \underline{\Pi_0(\mathcal{M})}$ *the sheaf of* $\theta_{\mathcal{M}}$*-modules of germs of solutions of the conformal system of algebraic equations* (16a) *and* (16b)*, whatever is* α*, and extended by the* "third order system" *at each point* x_0 *in* $\mathcal{M}$*,*
2. $\widehat{\Gamma}_2(\mathcal{M}) \subset \underline{\Pi_2(\mathcal{M})}$ *the sheaf of* $\theta_{\mathcal{M}}$*-modules of germs of differential sections of* $\Pi_2(\mathcal{M})$ *satisfying the* "characteristic manifold" *algebraic equations for the conformal system at each* $x_0 \in \mathcal{M}$*, and which inherits of the* "Lie groupoid structure" *from* $\Pi_2(\mathcal{M})$.

With the local diffeomorphisms $\wedge^k T^*\mathcal{M} \otimes_{\mathbb{R}} J_r(\mathcal{M}) \simeq (\{x_0\} \otimes_{\mathbb{R}} J_r(\mathcal{M})) \times (\wedge^k T^*_{x_0}\mathcal{M} \otimes_{\mathbb{R}} J_r(\mathcal{M}))$, and from the last section we deduce the theorems:

Theorem 2. *The differential sequences below, where in the differential operators* $\widehat{\boldsymbol{\mathfrak{D}}}_1$*,* $\widehat{\boldsymbol{\mathfrak{D}}}_2$*,* $\boldsymbol{\mathfrak{D}}_1$ *and* $\boldsymbol{\mathfrak{D}}_2$ *are* $\mathbb{R}$*-linear,*

$$0 \longrightarrow \Gamma_0 \xrightarrow{j_1} \Gamma_1 \xrightarrow{\boldsymbol{\mathfrak{D}}_1} \underline{T^*\mathcal{M}} \otimes_{\theta_{\mathcal{M}}} \underline{J_1(\mathcal{M})} \xrightarrow{\boldsymbol{\mathfrak{D}}_2} \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_{\mathcal{M}}} \underline{J_1(\mathcal{M})}\,,$$

$$0 \longrightarrow \widehat{\Gamma}_0 \xrightarrow{j_2} \widehat{\Gamma}_2 \xrightarrow{\widehat{\boldsymbol{\mathfrak{D}}}_1} \underline{T^*\mathcal{M}} \otimes_{\theta_{\mathcal{M}}} \underline{J_2(\mathcal{M})} \xrightarrow{\widehat{\boldsymbol{\mathfrak{D}}}_2} \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_{\mathcal{M}}} \underline{J_2(\mathcal{M})}$$

are exact.

Remark 6. In each of these theorems, the sheafs $\underline{\Pi_i(\mathcal{M})}$ $(i = 1, 2)$ in the source sheafs of the differential operators $\boldsymbol{\mathfrak{D}}_2$ and $\widehat{\boldsymbol{\mathfrak{D}}}_2$, must be taken into account in a similar way as for $\underline{J_1}$ in the corresponding sequence for the "c" system.

From all the previous theorems and the splitting (38) and the one defined by (29) we also deduce:

Theorem 3. *The following diagram*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_0 \arrow[r, "j_1"] \arrow[d] & \Gamma_1 \arrow[r, "\boldsymbol{\mathfrak{D}}_1"] \arrow[d, "\underline{b_0}"] & \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1(\mathcal{M})} \arrow[r, "\boldsymbol{\mathfrak{D}}_2"] \arrow[d, "\underline{b_1}"] & \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1(\mathcal{M})} \\
0 \arrow[r] & \widehat{\Gamma}_0 \arrow[r, "j_2"] & \widehat{\Gamma}_2 \arrow[r, "\widehat{\boldsymbol{\mathfrak{D}}}_1"] \arrow[d, "\underline{e_0}"] & \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_2(\mathcal{M})} \arrow[r, "\widehat{\boldsymbol{\mathfrak{D}}}_2"] \arrow[d, "\underline{e_1}"] & \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_2(\mathcal{M})} \\
0 \arrow[r] & \mathcal{S}_c^0 \arrow[r, "j_1"] & \mathcal{S}_c^1 \arrow[r, "D_{1,c}"] \arrow[d] & \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1} \arrow[r, "D_{2,c}"] \arrow[d] & \wedge^2 \underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1} \\
 & & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of exact sequences of sheafs of modules with $\mathbb{R}$-linear differential operators.

Remark 7. We do not have any relation of the kind $\underline{e_2} \circ \widehat{\boldsymbol{\mathfrak{D}}}_2 = D_{2,c} \circ \underline{e_1}$, where $\underline{e_2}$ would be a map from $\underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_2(\mathcal{M})}$ to $\underline{T^*\mathcal{M}} \otimes_{\theta_\mathcal{M}} \underline{J_1}$. Hence the commutativity in the latter diagram can't be "extended" on the right to give a right vertical sequence between the sheafs of 2-forms. Indeed, in considering the splitting (39) in the expression $d\widehat{\Omega}^i_J - \sum_{k=1}^n dx_0^k \wedge \widehat{\Omega}^i_{J+1_k}$, and setting $(r = 1, \cdots, n)$

$$\mu_0 = d\sigma_0 - \sum_{k=1}^n dx_0^k \wedge \sigma_k \, ,$$

$$\mu_r = d\sigma_r - \sum_{k=1}^n dx_0^k \wedge \sigma_{kr} \, ,$$

we obtain:

$$\begin{aligned}
d\widehat{\Omega}^i_J - \sum_{k=1}^n dx_0^k \wedge \widehat{\Omega}^i_{J+1_k} = d\Omega^i_J - \sum_{k=1}^n dx_0^k \wedge \Omega^i_{J+1_k} + \chi^i_j \mu_0 + \sum_{k=1}^n \chi^{i,k}_J \mu_k + \\
[d\chi^i_J - \sum_{k=1}^n \chi^i_{J+1_k} dx_0^k] \wedge \sigma_0 + \sum_{k,r=1}^n \chi^{i,k}_J dx_0^r \wedge \sigma_{kr} + \\
\sum_{k=1}^n [d\chi^{i,k}_J - \sum_{r=1}^n (\chi^{i,k}_{J+1_r} - \chi^i_J \delta^k_r) dx_0^r] \wedge \sigma_k \, .
\end{aligned}$$

Then, if the relations (35) are satisfied (which are also satisfied by the Ω^i_J with $|J| \leq 3$), together with the relations (37) and (22), we deduce

$$\widehat{\boldsymbol{\mathfrak{D}}}_2 \widehat{\Omega}^i_J = \boldsymbol{\mathfrak{D}}_2 \Omega^i_J + [d\chi^i_J - \sum_{k=1}^{n} \chi^i_{J+1_k} dx^k_0] \wedge \sigma_0 + \sum_{k=1}^{n} [d\chi^{i,k}_J - \sum_{r=1}^{n} (\chi^{i,k}_{J+1_r} - \chi^i_J \delta^k_r) dx^r_0] \wedge \sigma_k + \sum_{k,r=1}^{n} \chi^{i,k}_J dx^r_0 \wedge \vartheta_{kr}(\boldsymbol{c}^1_0, \sigma_J) + \chi^i_j \, D_{2,c}\sigma_0 + \sum_{k=1}^{n} \chi^{i,k}_J \, D_{2,c}\sigma_k \, .$$

In order to make the diagram commutative then we must set:

$$[d\chi^i_J - \sum_{k=1}^{n} \chi^i_{J+1_k} dx^k_0] \wedge \sigma_0 + \sum_{k,r=1}^{n} \chi^{i,k}_J dx^r_0 \wedge \vartheta_{kr}(\boldsymbol{c}^1_0, \sigma_J) + \sum_{k=1}^{n} [d\chi^{i,k}_J - \sum_{r=1}^{n} (\chi^{i,k}_{J+1_r} - \chi^i_J \delta^k_r) dx^r_0] \wedge \sigma_k = 0 \, .$$

It follows from the latter equations that

$$d\chi^i_J \wedge dx^1_0 \wedge \cdots \wedge dx^n_0 \wedge \sigma_0 \wedge \sigma_1 \wedge \cdots \wedge \sigma_n = 0 \, ,$$
$$d\chi^{i,k}_J \wedge dx^1_0 \wedge \cdots \wedge dx^n_0 \wedge \sigma_0 \wedge \sigma_1 \wedge \cdots \wedge \sigma_n = 0 \, ,$$

which shows clearly that $d\chi^{i,k}_J$ and $d\chi^i_J$ are defined on T^*J_1, and consequently the χ's depend on $\boldsymbol{c}^1_0$. But from the relations (40) the χ's also depend on $\hat{\boldsymbol{a}}^2_0$, which involves that a sequence of sheafs between the sheafs of 2-forms would exist only if a section from J_1 to $\boldsymbol{\mathfrak{P}}_2$ is given. Anyway in this case the bottom and the middle sequences of the last theorem would be merely isomorphic.

6. The unfolded space-time and the gravitation

Again, in view of physical interpretations, we put a spotlight on the tensor $\mathcal{B}$. In fact, we consider the relations (39) with $|J| = 0$ and the $\widehat{\Omega}^i$ as fields of *"tetrads"* or *"soldering 1-forms"*. Then we get a new metric ν of what we call the *"unfolded space-time"* defined at x_0 by:

$$\nu = \sum_{i,j=1}^{n} \omega_{ij}(x_0) \, \widehat{\Omega}^i(x_0) \otimes \widehat{\Omega}^j(x_0) \, ,$$
$$\widehat{\Omega}^i = \sum_{k=1}^{n} \hat{\kappa}^i_k(x_0) \, dx^k_0 \, ,$$
$$\hat{\kappa}^i_j = \kappa^i_j(x_0) + \chi^i(x_0, \hat{\boldsymbol{\tau}}^2_0) \, \mathcal{A}_j(x_0) + \sum_{k=1}^{n} \chi^{i,k}(x_0, \hat{\boldsymbol{\tau}}^2_0) \, \mathcal{B}_{j,k}(x_0) \, .$$

We consider the particular case for which the metric $\omega \equiv \mathrm{diag}[+, -, \cdots, -]$, the χ's are only depending on x_0 and $\kappa^i_j = \delta^i_j$, *e.g.* the deformation of ω is only due to the tensors $\mathcal{A}$ and $\mathcal{B}$. Thus, one has the general relation between ν and ω: $\nu = \omega +$ linear and quadratic terms in $\mathcal{A}$ and $\mathcal{B}$. Then from this metric ν, one can deduce the Riemann and Weyl curvature tensors of the *"unfolded space-time"*. One has a non-metrical theory for the gravitation in the gauge space-time, since clearly ν doesn't appear as a gravitational potential.

The space-time terminology we use is quite natural in the sense that one has simultaneously two types of space-time. The first one, which we call the *"underlying"* or *"substrat" space-time*, is endowed with the metric ω and is of constant scalar curvature k_0. It is the *"compars" space-time*, *e.g.* the space-time of physical or material rulers and watches, or merely of *"material bodies"* (*e.g.* electrically charged or magnetized, massive and featured by the weights (m, s) of the finite irreducible representations of the Poincaré Lie group). The other one, called the *"dispars"* or *"unfolded space-time"*, endowed with the metric ν, is defined for any scalar curvature and by the gauge potentials $\mathcal{A}$ and $\mathcal{B}$. It can be considered as the underlying space-time, deformed by the gauge potentials and the Weyl curvature does not necessarily vanish. Moreover, from a continuum mechanics of deformable bodies point of view, the metric ν can be interpreted as the tensor of deformation of the underlying space-time [11].

We are faced to a question: could this kind of deformation be interpreted as an inflation process in cosmology in which each occurrence (a tick-tock) of a creation or annihilation of a potential of local interaction would lead to a *"thermodynamic clock"* related to the unfolding in the sense of I. Prigogine (not a geometric clock, this latter being associated to the non *"topologically evolving"* substrat space-time !) and inducing a *"measurable"* time evolution without measurable time origin as a consequence ? Indeed, in such situation, clocks would be produced by the time and conversely ! Also, would the inflation be the evolution from the Poincaré Lie structure to "a" conformal one, and going from a physically homogeneous space-time (namely with constant Riemann curvature and a vanishing Weyl tensor and so *"rigid"*) to an inhomogeneous one (with any Weyl tensor) ? Coming from a vanishing Weyl tensor to a non-vanishing one, is in accordance with some Big-Bang concepts, but with a very different meaning of time and space-time than those that would be considered in the aforementioned context. In some way, the potentials $\mathcal{A}$ and $\mathcal{B}$ would produce a *"bifurcation"* of the space-time structure leading to a different concept of bifurcation than the one used in the case of non-linear ODE's. A counting of the bifurcations would be an evaluation of a particular time, *e.g.* an ordered set of agreements on events shared by observers, each one associated to a Lorentz frame.

In view of making easier computations for a relativistic action deduced from the metric tensor ν, we consider this metric in the *"weak fields limit"*, *e.g* the metric ν is linear in the tensors $\mathcal{A}$ and $\mathcal{B}$ and the quadratic terms are neglected. Furthermore, from the relations (23), we have the relations:

$$\partial_i \mathcal{A}_k - \partial_k \mathcal{A}_i = \mathcal{B}_{k,i} - \mathcal{B}_{i,k} = \mathcal{F}_{k,i} \,,$$

$$\partial_j \mathcal{B}_{k,i} - \partial_k \mathcal{B}_{j,i} \simeq k_0 (\omega_{ik} \mathcal{A}_j - \omega_{ij} \mathcal{A}_k) \,,$$

since the functions ρ take also small values in the weak fields limit. Therefore, we can write

$$\nu_{ij} = \omega_{ij} + \epsilon_{ij} \,,$$

where the ϵ_{ij} coefficients are small perturbations of the metric ω and defined from $\mathcal{A}$ and $\mathcal{B}$ by the formula:

$$\epsilon_{ij} \simeq \sum_{k=1}^{n} \chi^k \left(\omega_{kj} \mathcal{A}_i + \omega_{ki} \mathcal{A}_j \right) + \sum_{k,h=1}^{n} \chi^{k,h} \left(\omega_{kj} \mathcal{B}_{i,h} + \omega_{ki} \mathcal{B}_{j,h} \right) \,.$$

Then, let i be a differential map $i : s \in [0, \ell] \subset \mathbb{R} \longrightarrow i(s) = x_0 \in \mathcal{M}$. We define the relativistic action S_1 by:

$$S_1 = \int_0^\ell \sqrt{\nu(u(s), u(s))}\, ds \equiv \int_0^\ell \sqrt{2 L_\nu}\, ds \,,$$

where $u(s) \equiv di(s)/ds$. We also take the tensors χ as depending on s. The Euler-Lagrange equations for the Lagrangian density $\sqrt{L_\nu}$ are not independent because $\sqrt{L_\nu}$ is a homogeneous function of degree 1 and thus satisfies an additional homogeneous differential equation. Then, it is well-known that the variational problem for S_1 is equivalent to consider the variation of the action S_2 defined by

$$S_2 = \int_0^\ell \nu(u(s), u(s))\, ds \equiv \int_0^\ell 2 L_\nu \, ds \,,$$

but constrained by the condition $2L_\nu = 1$. In this case, it shows that $2L_\nu$ must be considered, firstly, as a Lagrange multiplier, and secondly, its explicit expression with respect to u will appear only in the variational calculus. In the weak fields limit and with $\|u\|^2 \equiv \omega(u, u) = \sum_{i,j=1}^n \omega_{ij} u^i u^j = (u^1)^2 - (u^2)^2 - \cdots - (u^n)^2$, we obtain:

$$L_\nu = \frac{1}{2} \|u\|^2 + \sum_{j,k=1}^{n} \omega_{kj}\, \chi^k\, u^j \,.\, \sum_{i=1}^{n} \mathcal{A}_i\, u^i + \sum_{j,k,h=1}^{n} \chi^{k,h}\, \omega_{kj}\, u^j \,.\, \sum_{i=1}^{n} u^i\, \mathcal{B}_{i,h} \,. \tag{41}$$

But also, if $0 < \|u\|^2 < +\infty$ and setting $\|u\| \equiv 1/\gamma$, we have:

$$\sqrt{2L_\nu} \simeq \gamma^{-1} + \sum_{j,k=1}^{n} \omega_{kj}\, \chi^k \,(\gamma u^j) \,.\, \sum_{i=1}^{n} \mathcal{A}_i\, u^i + \sum_{j,k,h=1}^{n} \chi^{k,h}\, \omega_{kj}\, (\gamma u^j) \,.\, \sum_{i=1}^{n} u^i\, \mathcal{B}_{i,h} \,. \tag{42}$$

From the latter relation, we can deduce a few physical consequences among a lot of other particular ones. On one hand, if we assume that

$$\sum_{j,k=1}^{n} \omega_{kj} \chi^{k,h} U^j \equiv \xi^h = cst\,, \tag{43a}$$

$$\sum_{k,j=1}^{n} \omega_{kj} \chi^{k} U^j \equiv \xi_0 = cst\,, \tag{43b}$$

where $U \equiv \gamma u$, then we recover in (42), up to some suitable constants, the Lagrangian density for a particle, with the *n-velocity vector* U ($\|U\|^2 = 1$), embedded in an external electromagnetic field. But also from the relation (43b) we find a *"Thomas precession"* if the tensor (χ^k) is ascribed to a *"polarization n-vector"* [3, p.270] *"dressing"* the particle (ex.: the spin of an electron).

On the other hand, we have also a physical interpretation in the framework of the magnetoelectric interaction phenomena in crystals as described in chapter 2. Assuming the symmetric part of $\mathcal{B}$ is vanishing only, $\xi^h = cst \neq 0$ and the relation (43b) satisfied, then we recover in (42) the Lagrangian density ${}^2\mathcal{L}$ of E. Asher [2] for magnetoelectric phenomena. Moreover the relation (43a) with $\xi^h = cst \neq 0$ is precisely the condition for to get a *"generalized Thomas precession"* in magnetoelectric crystals. This generalized precession could give a possible origin for the creation of anyons in high-T_c superconductors [21, 20] and might be an alternative to Chern-Simon theory. In this situation the tensor $(\chi^{k,h})$ is a polarization tensor of a crystal and the particle is *"dressed"* with this kind of polarization (not polarized by ...). The condition $\xi^h = cst$ is due to the relativistic symmetry of the crystal. Anyway since we have an "empty" space-time with *a priori* no relativistic crystalline structures, the crystalline relativistic symmetry can come from crystalline (periodic in space-time, namely waves) electromagnetic or $\mathcal{B}$ fields dressing the particle, *e.g.* dressed by the crystalline part of the external Faraday tensor fields or the more general $\mathcal{B}$ fields. Hence, this last interaction is not really due to the electromagnetic or "$\mathcal{B}$" interaction but to the need for a local relativistic symmetry conservation.

Another strange situation would be $\xi^h \equiv x_0^h = i^h(s)$ meaning $(\chi^{k,h})$ would be a magnetic moment, and if in addition the symmetric part of $\mathcal{B}$ vanishes, then the expression

$$\sum_{i,h=1}^{n} (x_0^h\, u^i - u^h\, x_0^i)\, \mathcal{F}_{i,h}$$

would describe an interaction of an electromagnetic fields with a magnetic moment as in the Larmor precession. Let us notice to finish that we have also to consider in addition the Lagrangian density (27) for the dynamics of $\mathcal{A}$ and $\mathcal{B}$.

More generally, the Euler-Lagrange equations associated to S_2 would be a system of geodesic equations with Riemann-Christoffel symbols associated to ν and such that (with $\nu^{ij} \simeq \omega^{ij}$ at first order and assuming the χ's being constants)

$$\frac{du^r}{ds} = \frac{1}{2}\sum_{j,k=1}^{n} \mathcal{P}^r_{jk}\, u^j u^k + \sum_{i,h,s,t=1}^{n} \left(\xi_0\, \omega^{rs}\, \mathcal{F}_{i,s} + k_0\, \xi^h \left(\delta^r_h\, \mathcal{A}_i - \mathcal{A}_t\, \omega^{rt}\, \omega_{hi}\right)\right) u^i\,, \quad (44)$$

with

$$\mathcal{P}^r_{jk} = \chi^r(\partial_k \mathcal{A}_j + \partial_j \mathcal{A}_k) + \sum_{h=1}^{n} \chi^{r,h}\left(\partial_j \mathcal{B}_{k,h} + \partial_k \mathcal{B}_{j,h}\right)\,,$$

and $\mathcal{A}$ and $\mathcal{B}$ satisfying the first and second sets of differential equations. Then the tensor $\mathcal{P}$ would be associated to gravitation fields, providing other physical interpretations for the tensors χ.

Nevertheless, the equations (44) are deduced without the conditions (43a) and (43b). Taking them into account would lead to a modification of the action S_2 by adding Lagrange multipliers λ_0 and λ_k ($k = 1, \cdots, n$) in the Lagrangian density definition, and changing the variable of integration we can define a new action:

$$S^2_\tau = \int_0^\ell \left\{ m + \sum_{i=1}^{n} \epsilon_i(\xi_0, \xi^h)\, U^i - \left(\lambda_0 \xi_0 + \sum_{k=1}^{n} \lambda_k \xi^k\right)\right\} d\tau\,,$$

with the measure $d\tau \equiv ds/\gamma$ on the hyperboloid $H(1, n-1)$, and $\epsilon_i(\xi_0, \xi^h) \equiv \sum_{k=1} \epsilon_{ki} U^k$ with relations (43). The associated Euler-Lagrange equations would be analogous to (44) with U instead of u but with additional terms coming from the precession. Moreover, since we have the constraint $\|U\|^2 = 1$, we need a new Lagrangian multiplier denoted by m (> 0) ! That also means we do computations on the projective spaces $H(1, n-1)$ of the tangent spaces and make variational calculus with an induced metric $\nu_P = \nu(P_*, P_*)$ with P a particular projector, and a measure $P^*(ds) = d\tau$. From this view point, the Lagrange multipliers appear to be non-homogeneous coordinates of these projective spaces. Alternatively from (42) we can take the action

$$S^1_\tau \equiv \int_0^\ell \left\{ m + \xi_0 \,.\sum_{i=1}^{n} \mathcal{A}_i\, U^i + \xi^h \,.\sum_{i=1}^{n} U^i\, \mathcal{B}_{i,h} - \left(\lambda_0 \xi_0 + \sum_{k=1}^{n} \lambda_k \xi^k\right)\right\} d\tau\,.$$

Then the variational calculus would also lead to additional precession equations giving torsion as mentioned in the comments about the equations (2). Again the torsion is not related to the unification but to parallel transports on manifolds which is a well-known geometrical fact [5].

Remark 8. All functions and applications are defined at x_0 and not at $x \in U_{x_0}$, meaning we must write Taylor series at x_0 to deduce their values at x. In other words, physical evaluations of $X \equiv x - x_0$ must be done, and all the variables at x (χ, $\mathcal{A}$, etc ...) would be functions of x_0 and X. But X is not physically evaluable, since we would be able to travel in time to go to x and come back to x_0, or more generally to turn "freely" in space-time around x_0. Hence we can only do measurements at x_0 of light rays coming from x. Then we must build up, from the light wave vectors denoted k, the position of x. It follows that we would have to find a sheaf map from a neighborhood of x_0 to a neighborhood of the origin of the tangent space $T_{x_0}\mathcal{M}$ containing the k's. This can be only done locally, involving we shall call position vectors what would be in fact vectors in the tangent space and not the X's. That is an equivalence principle. Then wave functions, for instance, would depend, at least, on x_0 and tensorial variables. In particular, this is expressed in the series s for the functions α when inverting x and x_0.

7. Conclusion

In fact in this work, using the Pfaff systems theory instead of the Kumpera-Malgrange-Spencer theory of Lie equations, we studied the formal solutions of the conformal Lie system with respect to the Poincaré one. More precisely, we determined the difference between these two sets of formal solutions. It is described by a "relative" set of no Lie PDE's, namely the "c" system. We studied these two Lie systems of Lie equations because of their occurrences in physics and particularly in electromagnetism as well as in Einsteinian relativity. We just made the assumption that the *"substrat space-times"* would be equivariant with respect to the conformal pseudogroup and setting its Riemann scalar curvature to be a constant k_0. Then, we built up differential complexes and tried to give with some theorems, interpretations in physics of the various tensors coming from the relative complex. This was only a "classical approach" and quantization didn't seem to appear. Nevertheless a deeper analysis of the latter conformal actions, which are of Polyakov type in dimension 4 or also a 1-acyclic cocycle, in the framework of the A. Connes non-commutative geometry would work out likely a quantization of the space of leafs of the hyperboloid space of the 4-vectors U. Alternatively another approach, with the concept of time operator and the signature of the metric (which might change in this model), can be consider in the framework of Kolmogorov flows [17].

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